Day 7 - AM

Thm: (for 2x2)

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $\det A = ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

if $ad - bc = 0$, then A is not invertible.

A square matrix A is said to be diagonalizable if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D.

Ex: Diagonalize the following matrix, if possible

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

(find invertible P and diagonal D such that $A = PDP^{-1}$)

Step 1: Find eigenvalues of A.

Thm: $\lambda$ is an eigenvalue of nxn matrix A iff $\det(A - \lambda I) = 0$.

$$0 = \det(A - \lambda I) = -\lambda^3 - 3\lambda^2 + 4$$
$$= -(\lambda - 1)(\lambda + 2)^2$$

Eigenvalues are $\lambda = 1$ and $\lambda = -2$.

Step 2: Find linearly independent eigenvectors of A (we need 3 since A is 3x3)

if $\lambda = 1$ $A - \lambda I = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix}$

now reduce the augmented matrix

$$\left[\begin{array}{ccc|c} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$a - c = 0$

$b + c = 0$

$c = a = -b$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} c \\ -c \\ c \end{bmatrix} = c \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

basis for $\lambda = 1$ : $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

do the same for $\lambda = -2$, get basis : $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\checkmark$: these 3 are linearly independent

Step 3: Construct P from basis vectors (order not important)

$$P = [v_1 \ v_2 \ v_3] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Step 4: Construct D from corresponding eigenvalues (order must match the columns

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Lastly, check your work: $A = PDP^{-1} \Rightarrow AP = PD$

you check: $AP = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$

$$PD = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \checkmark$$

Ex: Diagonalize the following matrix, if possible

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

$0 = \det(A - \lambda I) = -\lambda^3 - 3\lambda^2 + 4 = -(\lambda - 1)(\lambda + 2)^2$ so $\lambda = 1, -2$

basis for $\lambda = 1$: $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ basis for $\lambda = -2$: $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

We cannot construct a basis of $\mathbb{R}^3$ with only 2 vectors, thus A is not diagonizable.

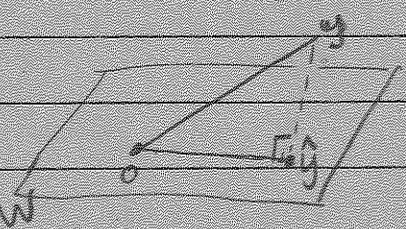
<u>dot product</u> or <u>inner product</u> of u and v

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad \text{then}$$

$$u \cdot v = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \dots + u_nv_n$$

two vectors u and v in $\mathbb{R}^n$ are <u>orthogonal</u> (to each other) if $u \cdot v = 0$.

(Geometrically, orthogonal vectors are perpendicular vectors)




$\hat{y} = \text{proj}_W y$ is the orthogonal projection of y onto W (a subspace of $\mathbb{R}^n$)

if $\{u_1, u_2, \dots, u_p\}$ is an orthogonal basis of W

then $\hat{y} = \text{proj}_W y = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \dots + \frac{y \cdot u_p}{u_p \cdot u_p} u_p$

dot products

<u>Ex:</u> $y = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix} \quad u_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \quad u_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

let W a subspace of $\mathbb{R}^n$, $W = \text{span}\{u_1, u_2\}$

$u_1 \cdot u_2 = 5(1) + -2(2) + 1(-1) = 0$ thus $\{u_1, u_2\}$ is an orthogonal basis for W then

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{15}{30}\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + \frac{-21}{6}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \text{ reduce and simplify } = \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix}$$

## Symmetric Matrices $A^T = A$

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & 8 \\ 0 & 8 & -7 \end{bmatrix}, \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

## The Spectral Theorem for Symmetric Matrices

An $n \times n$ symmetric matrix $A$ has the following properties:

a) $A$ has $n$ real eigenvalues, counting multiplicities

b) The dimension of the eigenspace for each eigenvalue $\lambda$ equals the multiplicity of $\lambda$ as a root of the characteristic equation.

c) The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.

d) $A$ is orthogonally diagonalizable.